On gravitational collapse against a cosmological background

1970 J. Phys. A: Gen. Phys. 3 653

(http://iopscience.iop.org/0022-3689/3/6/007)

J. Phys. A: Gen. Phys., 1970, Vol. 3. Printed in Great Britain

On gravitational collapse against a cosmological background

A. BARNES

Department of Mathematics, Imperial College of Science and Technology, London SW7, England

MS. received 11*th June* 1970

Abstract. The propagation of light in a spherically symmetric dust distribution is investigated. A sufficient condition for light to escape from a collapsing region to an expanding region of the system is obtained. This corrects the 1966 result of Raychaudhuri. It is also shown that, contrary to the conclusion drawn by Som in 1968, no event-horizon other than that associated with the Schwarzschild radius occurs. These results are illustrated by some numerical calculations for particular models.

1. Introduction

The gravitational frequency shift of light radially transmitted from a static source to a static observer in the Schwarzschild exterior field has long been known (Schwarzschild 1916). Synge (1966) obtained the corresponding result for non-radial light waves. More recently the frequency shift of light radiated from a pressure-free collapsing sphere has been considered by Faulkner *et al.* (1964), and also by Banerjee (1966 a), who obtained an analytic formula. The generalization of these results for non-radial light rays was given by Jaffe (1969).

The additional effects of cosmological expansion and of the gravitational field of matter between the source and the observer have been considered by Raychaudhuri (1966) and by Banerjee (1966 b) and by Som (1968). If there are no discontinuities in the density, Raychaudhuri concluded that light from the body would be trapped as soon as collapse sets in, but this will be shown to be incorrect. Instead, so long as light is not transmitted through any matter that has contracted to a radius $R < 2M(r)$, where $M(r)$ is the gravitational mass of matter within the co-moving coordinate radius r, it will escape. This result holds even if there are discontinuities in density. We note that the 2-surfaces (R = constant $< 2M(r)$, r = constant) are closed trapped surfaces (Penrose 1965). Thus it appears that no essentially new features occur when the expansion of the universe is included in the analysis.

2. The equations governing a pressure-free system

In general relativity the metric of a spherically symmetric system may be written (Landau and Lifshitz 1962) in terms of co-moving coordinates r, θ, ϕ and t in the form

$$ds^2 = dt^2 - (R'/\Gamma)^2\, dr^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

with

$$\dot{R}^2 = \Gamma^2 - 1 + 2M(r)/R \tag{2}$$

and

$$M' = 4\pi\rho R^2 R' \tag{3}$$

where Γ and M are arbitrary functions of r, ρ is the energy density of matter and prime and dot symbols denote differentiations with respect to r and t respectively.

Differentiating equation (2) with respect to t we obtain

$$\ddot{R} = -M/R^2. \tag{4}$$

Equations (2)–(4) are identical in form with the Newtonian equations and so we interpret M as the active gravitational mass within coordinate radius r. We will consider only matter with positive gravitational mass since no matter with negative mass has yet been observed.

The proper distance s from the centre to radius r is a strictly increasing function of r and, since s is given by

$$s = \int_0^r (R'/\Gamma)\,\mathrm{d}r$$

we see that $R'/\Gamma \geqslant 0$. We deduce from (1) that Γ is the derivative of R with respect to proper distance in the radial direction, and so where $\Gamma < 0$ the gravitational field is so strong that it begins to close the space: i.e. on hypersurfaces of constant t, the area of the 2-surfaces given by r constant decreases as r increases.

We will assume that $r = 0$ is a point, i.e. $R(0, t) = 0$, and from (2) we see that $M(0) = 0$ and $\Gamma(0) = 1$.

To simplify the analysis below, we write $\Gamma^2 - 1 = f$. Integration of equation (2) then yields

$$t = \frac{2M\{(x+x^2)^{1/2} - \sinh^{-1}(x)^{1/2}\}}{f^{3/2}} \qquad \text{for } f > 0 \tag{5}$$

$$t = \frac{2R^{3/2}}{3(2M)^{1/2}} \qquad \text{for } f = 0 \tag{6}$$

$$t = \frac{2M\{\sin^{-1}(x)^{1/2} - (x-x^2)^{1/2}\}}{(-f)^{3/2}} \qquad \text{for } f < 0 \text{ and } \dot{R} > 0 \tag{7a}$$

$$t = \frac{2M\{\pi - \sin^{-1}(x)^{1/2} + (x-x^2)^{1/2}\}}{(-f)^{3/2}} \qquad \text{for } f < 0 \text{ and } \dot{R} < 0 \tag{7b}$$

where $x = |f|R/2M$ and $\sin^{-1}$ takes its principal value. We have chosen the arbitrary function of integration so that the system originates from a singularity at time $t = 0$, i.e. $R(r, 0) = 0$ for all r.

When M is constant, we see from equation (3) that $\rho = 0$, and hence by Birkhoff's theorem that (1) must be the exterior Schwarzschild form. In fact, the coordinate transformations

$$\mathrm{d}T = \mathrm{d}t/\Gamma + \dot{R}\,\mathrm{d}r/(1-2M/R)\Gamma$$

and

$$R_{\mathrm{Sch}} = R(r, t)$$

transform (1) to the usual Schwarzschild form.

The equations (1)–(7) given above are equivalent to those of Raychaudhuri (1966), except that he used (7*a*) for both positive and negative $\dot{R}$ which is incorrect.

We note from (7) that matter with $f < 0$ expands to a maximum radius $2M/(-f)$ at time $T = \pi M/(-f)^{3/2}$ and then contracts to a singularity at time $2T$. We will assume that after this time the matter coalesces to form a point-mass at $R = 0$. From (6) we see that matter with $f = 0$ expands to infinity with $R = (9Mt^2/2)^{1/3}$, whilst from (5) it follows that shells with $f > 0$ expand to infinity with $R \sim f^{1/2}t$ for large t.

The particular system described by the model depends, of course, on f (or Γ) and M. For instance, if $M = \beta r^3$ and $f = kr^2$, we obtain one of the Friedmann universes, whereas Raychaudhuri's example, namely $M = \beta r^3$, $f = -r^2\{2\exp(-r/r_0)-1\}$, represents a universe that for large r is asymptotically a negatively curved Friedmann universe containing a region ($r < r_0 \ln 2$) that ultimately contracts.

3. Restrictions on the solution

Γ and M are not completely arbitrary since the first and second fundamental forms of any hypersurface, r = constant = b, must be the same whether the hypersurface is regarded as embedded in the region $r < b$ or $r > b$. This ensures that a system of admissible coordinates, i.e. one in which the metric tensor is C^1 and piecewise C^3, exists (Robson 1968, Misner and Sharp 1964). The continuity of the first fundamental form implies that $R(r, t)$, and hence $\dot{R}$, are continuous. Consequently, from (2) we see that Γ^2 and M are continuous. The continuity of the second fundamental form ψ, where

$$\psi = R\Gamma(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)$$

shows that Γ itself is continuous. Continuity of M implies that surface distributions of mass do not exist.

We note that $R'/\Gamma \geqslant 0$ cannot hold for all t if R' changes sign, and so we must find conditions on M and f that ensure that this does not occur. We need the following relations:

$$R' = \left(\frac{M'}{M} - \frac{f'}{f}\right)R - \dot{R}t\left(\frac{M'}{M} - \frac{3f'}{2f}\right) \qquad \text{for } f \neq 0 \tag{9a}$$

and

$$R' = \frac{M'R}{3M} + \frac{f'R^2}{10M} \qquad \text{for } f = 0 \tag{9b}$$

which follow from (2), (5), (6) and (7); and

$$\dot{R}t = 2R/3 \qquad \text{for } f = 0 \tag{10a}$$

$$\dot{R}t < 2R/3 \qquad \text{for } f < 0 \tag{10b}$$

and

$$2R/3 < \dot{R}t < R \qquad \text{for } f > 0 \tag{10c}$$

which follow from (2), (4) and (6).

It will now be shown that $R' > 0$ if and only if

$$M' \geqslant 0 \quad \text{and} \quad \frac{M'}{M} - \frac{3f'}{2f} \geqslant 0 \qquad \text{for } f < 0 \tag{11a}$$

$$M' \geqslant 0 \quad \text{and} \quad f' \geqslant 0 \qquad \text{for } f \geqslant 0 \tag{11b}$$

but where f' and M' are not simultaneously zero.

That these conditions are sufficient for $f = 0$ can be seen immediately from (9*b*). For $f > 0$, using (9*a*) and (10*c*) we have

$$R' > (\tfrac{3}{2}\dot{R}t - R)f'/f > 0.$$

For $f < 0$, a similar argument, using (9*a*) and (10*b*), shows that the conditions (11*a*) are sufficient. The condition $M' \geqslant 0$ is necessary since from (9), and the fact that

$\dot{R}t \sim \frac{2}{3}R$ for small t, we see that $R' \sim M'R/3M$. For $f \geqslant 0$ the necessity of the second condition $f' \geqslant 0$ follows from the asymptotic behaviour of R' for large t, namely $R' \sim f't/2f^{1/2}$ for $f > 0$ and $R' \sim f'R^2/10M$ for $f = 0$. If $f < 0$, we note that collapse back to a singularity occurs after time $T = 2\pi M/(-f)^{3/2}$ and, since $\mathrm{d}T/\mathrm{d}r \geqslant 0$ (for otherwise R would decrease outwards at some stage of the collapse sufficiently near the final singularity), the second of the conditions (11*a*) follows directly.

To obtain necessary and sufficient conditions for $R' < 0$ we need only reverse the inequalities in (11).

If the conditions (11) do not hold then at some stage of the motion collisions take place between neighbouring shells and we can no longer use co-moving coordinates. We could try to follow the motion after this point using non-co-moving coordinates, but from (3), we note that if $M' \neq 0$, $R' = 0$ implies that ρ is infinite, and so we would expect the zero-pressure assumption to be violated. We also note that the conditions (11) imply that ρ is never negative at any stage of the motion.

We now consider the situation on hypersurfaces with $\Gamma = 0$. If Γ^2 has a zero of order n and the metric (1) is non-singular, then R' must have a (permanent) zero of order $\frac{1}{2}n$ and if ρ is to be non-singular then so must M'. From equation (9*a*) for consistency we must have $n \geqslant 2$. However, so long as these conditions are satisfied there is nothing singular about the hypersurface $\Gamma = 0$ and it cannot be an 'impenetrable barrier' as Bondi (1947) stated. For example, in the positively curved Friedmann universe we have $\Gamma = \cos r$, $f = -\sin^2 r$, $M = \beta \sin^3 r$ and $R = S(t)\sin r$ but the hypersurface $r = \pi/2$ is in no way singular.

4. The propagation of light

We consider only light emitted radially, i.e. $\theta = \phi =$ constant. For light emitted outwards, that is r increasing along the ray, we have from (1), putting $\mathrm{d}s = 0$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\Gamma}{R'} \tag{12}$$

and using (2),

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \Gamma + \dot{R} = \Gamma \pm \left(\Gamma^2 - 1 + \frac{2M}{R}\right)^{1/2} \tag{13}$$

the positive sign arising when $\dot{R} > 0$ and negative sign when $\dot{R} < 0$.

We now restrict attention to systems with $f < 0$ for $r < a$ and $f \geqslant 0$ for $r \geqslant a$, where a is a constant. This class comprises systems which contain a region ($r < a$) that ultimately collapses, surrounded by a region which expands to infinity. We will also assume that M and f are bounded, except possibly as r tends to infinity. This rules out infinite masses in finite regions of spacetime, and is also a restriction on the coordinate systems we use, since it rules out those systems that cover the whole of spacetime with a finite range of values of r.

Consider first the case where $\Gamma < 0$ for some $r < a$. Since $\Gamma \geqslant 1$ for $r \geqslant a$, it follows that there is a neck $\Gamma = 0$ separating regions with $\Gamma < 0$ from the expanding background. It follows immediately from (13) that, if a ray from the region $r < a$ reaches the neck after collapse has set in, then it can never escape to the expanding region since $\mathrm{d}R/\mathrm{d}t < 0$. Whether it escapes if it reaches the neck whilst it is expanding will depend on the details of the particular model.

If $\Gamma > 0$ and $\dot{R} < 0$, we see from (13) that $\mathrm{d}R/\mathrm{d}t \leqslant 0$ whenever $2M/R \geqslant 1$. Since $M' \geqslant 0$, it follows that $\mathrm{d}(2M/R)/\mathrm{d}t \geqslant 0$, and so if at any point on the ray the condition $2M/R \geqslant 1$ holds, then it will continue to hold along the ray.

We have seen that the light cannot escape from collapsing regions once they have passed within their own Schwarzschild radius.

However if, for $r < a$, $2M/R < \alpha < 1$ along the ray, we have from (13)

$$\frac{\mathrm{d}R}{\mathrm{d}t} > \Gamma - (\Gamma^2 - 1 + \alpha)^{1/2} > 1 - \alpha^{1/2} > 0$$

but for the shell $r = a$ ($f(a) = 0$) it follows from (2) and (16) that

$$\dot{R}(a, t) = (4M/3t)^{1/3} \to 0 \qquad \text{as} \qquad t \to \infty.$$

Thus the light ray must eventually overtake the shell $r = a$ and reach the expanding region. The condition $2M/R < \alpha < 1$ is not very useful in practice, since, in order

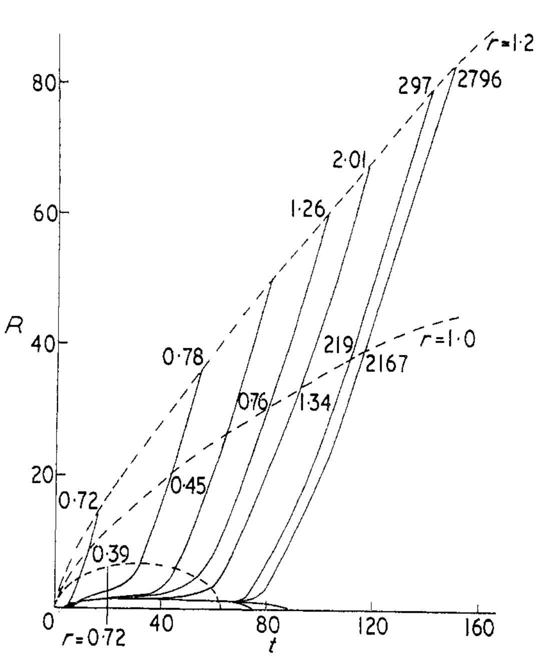

Figure 1. The results of numerical integrations for light rays in Raychaudhuri's model ($f = -r^2\{2\exp(-r/r_0) - 1\}$ and $M = r^3$ where $r_0 = 1/\ln 2$ and hence $f(1) = 0$) are shown. For small r a singularity forms after a time $t = 6{\cdot}283185$ but light emitted from the centre is not trapped until after $t = 6{\cdot}283183$. At $r = 0{\cdot}72$ the respective values are 63·38 and 62·86. The broken lines represent the history of shells with r constant and the numbers given at the intersections are the values of the red-shift z of light from $r = 0{\cdot}72$ and received by an observer moving with the matter.

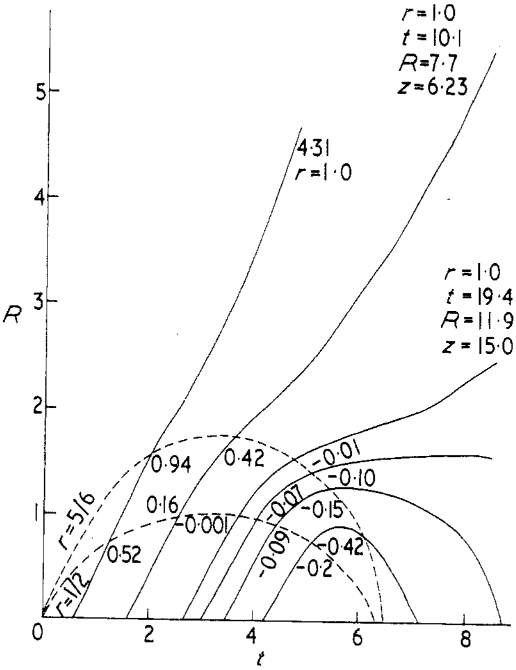

Figure 2. As for figure 1 except that here $f = -r^2 + r^{20}$. This model has a smaller minimum value of f ($-0{\cdot}854$ compared with $-0{\cdot}116$ in Raychaudhuri's example) and light from the centre is trapped after time $t = 3{\cdot}04$. This is earlier than the point of maximum expansion which occurs after a time $t = 3{\cdot}141593$. The red-shift values given refer to light emitted from the centre.

to see if it is satisfied, one would need to integrate (numerically) either (12) or (13). However, a sufficient condition for a ray to escape is clearly $2M(a)/R_{em} < 1$ when the suffix 'em' denotes the value at the point of emission of the light ray. This last result seems to contradict that of Raychaudhuri (1966) who states that a ray would be trapped if it met collapsing matter at any point along its path. We illustrate this point with numerical integrations (see figures 1–3) for several models including that

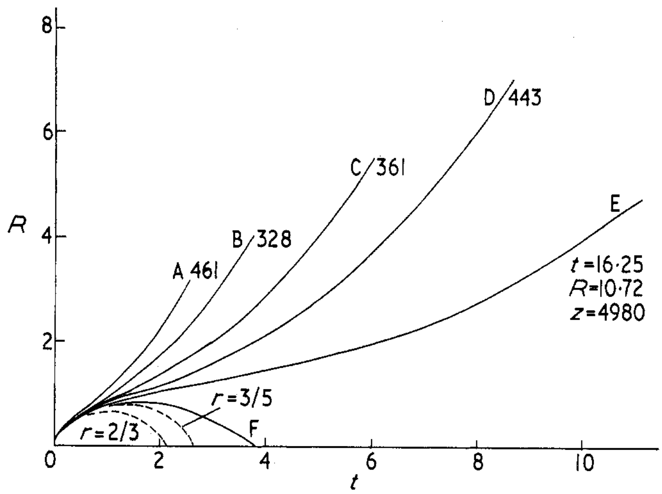

Figure 3. This model contains a region with $R' < 0$ and a neck at $r = 2/3$. Light emitted from shells with r less than about 0·55 can never escape and the cut-off at the neck occurs at about time $t = 0{\cdot}36$. Results are shown for several rays emitted from $r = 0{\cdot}6$. Rays emitted from this radius after a time $t = 0{\cdot}026$ are trapped. The rays labelled by A, B, C, D, E and F are emitted at times 0·001, 0·004, 0·011, 0·019, 0·024 and 0·044 respectively. The explicit forms of M and F are:

$f = -\sin^2 3\pi r/2,$	$M = \sin^3 3\pi r/2,$	for $r \leqslant 1/3$
$f = -(2+\sin 3\pi r/2)/3,$	$M = (2-\cos 3\pi r)/3,$	for $1/3 < r \leqslant 1/2$
$f = -(2-\cos 3\pi r/2)/3,$	$M = (2-\cos 3\pi r)/3,$	for $1/2 < r \leqslant 2/3$
and		
$f = \cos 3\pi r/2,$	$M = (2-\cos 3\pi r)/3+(r-2/3)^3,$	for $2/3 < r \leqslant 1.$

given by Raychaudhuri. A ray emitted from a shell that is expanding may eventually meet collapsing matter which is within its Schwarzschild radius and hence the ray becomes trapped. For an example of this latter phenomenon one need only consider a section of a positively curved Friedmann universe separated from the expanding universe by a region of empty space.

We now consider the question of whether an event-horizon other than that associated with the Schwarzschild radius develops. From (9*a*), (10*c*) and (12), we obtain for $f > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} > \frac{(1+f)^{1/2}}{(M'/3M+f'/2f)R}. \tag{14}$$

From (5) we have

$$\frac{f^{1/2}\,t}{R} = \left(1+\frac{1}{x}\right)^{1/2} - \frac{\sinh^{-1} x^{1/2}}{x} \tag{15}$$

and so, unless $x \to 0$ along the ray, there exists a positive β such that

$$R < \beta f^{1/2} t.$$

If, however, $x \to 0$ along the ray, by expanding (15) as a power series in x, it follows that for some $\beta_1 > 0$ and sufficiently large t

$$R < \beta_1 M^{1/2} t^{2/3}.$$

From (14) we have

$$\int_{r_{\mathrm{em}}}^{r} \frac{(M'/3M + f'/2f)M^{1/2}\,\mathrm{d}r}{(1+f)^{1/2}} > \int_{t_{\mathrm{em}}}^{t} \frac{\mathrm{d}t}{\beta_1 t^{2/3}}$$

and since the right-hand integral tends to infinity as $t \to \infty$ and the integrand on the left-hand side is non-zero and is bounded, except possibly for large r, it follows that $r \to \infty$ as $t \to \infty$. A similar argument proves that $r \to \infty$ as $t \to \infty$ if $R < \beta f^{1/2} t$. Thus no event-horizon develops in the region $r > a$. This seems to contradict a result of Som (1968) who considered the special case where the collapsing body is separated from the background by a region of empty space. We merely note than an event-horizon would occur if a positive cosmological constant were introduced.

A further possibility remains: a ray may approach asymptotically the coordinate radius a. For such a ray $\mathrm{d}R/\mathrm{d}t$ must tend to zero, and so it follows from (12) that $R \to 2M(a)$ along the ray in such a way that for large t the ray is passing through collapsing matter. We now show there is only one such ray (for given θ, ϕ and r_{em}) and rays which leave at later times cannot escape, whereas those emitted earlier escape in a finite time.

If there were two such rays then all rays emitted at intermediate times would have this property, otherwise they would cross one or other of the two rays and the future null direction at this event would not be well-defined. If we label two such rays by I and II, II being later than I, we have $r_{\mathrm{I}}(t) > r_{\mathrm{II}}(t)$ and hence (as $R' > 0$) $R_{\mathrm{I}}(t) > R_{\mathrm{II}}(t)$. Since for both rays $R(t) \to 2M(a)$ as $t \to \infty$, it follows that eventually

$$\mathrm{d}R_{\mathrm{I}}(t)/\mathrm{d}t < \mathrm{d}R_{\mathrm{II}}(t)/\mathrm{d}t$$

and since this holds for all rays between I and II, then $(\mathrm{d}R/\mathrm{d}t)' < 0$.

However, as $t \to \infty$, $r_{\mathrm{I}}(t) \to a$, $R_{\mathrm{I}}(t) \to 2M(a)$ and $f\{r(t)\} \to 0$ and so eventually $-2M/fR > 1$. Now, from (9*a*) and (11*a*), it follows that by using $\dot{R} < 0$ we obtain

$$(-f)\left(\frac{2M}{-fR}\right)' = \left(\frac{2M}{R}\right)' + \frac{2Mf'}{R(-f)} < 0$$

and so

$$\left(\frac{2M}{R} + f\right)' < 0.$$

Using (13) we have

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)' = \{(1+f)^{1/2}\}' - \left\{\left(\frac{f+2M}{R}\right)^{1/2}\right\}' > 0$$

and since this contradicts the condition above, there can be only one such ray. In the particular case of a collapsing body surrounded by a region of empty space, this ray is emitted as the body passes through its Schwarzschild radius.

5. The frequency shift

If u^a is the four-velocity of the matter and k^a the four-momentum of the radiation, then the fractional shift in frequency z of radiation emitted from the matter and received by an observer moving with the matter is given by

$$1+z = \frac{(k^a u_a)_{\text{emitted}}}{(k^a u_a)_{\text{observed}}} = \frac{k^4{}_{\text{emitted}}}{k^4{}_{\text{observed}}}.$$

Since k^a is given by

$$k^a k_a = 0, \qquad k^a{}_{;b}\, k^b = 0$$

we obtain

$$k^{4\prime} \pm (k^4 R'/\Gamma)^{\cdot} = 0 \tag{16a}$$

and so

$$\frac{\mathrm{d}k^4}{\mathrm{d}r} = \mp \frac{k^4 \dot{R}'}{\Gamma} \tag{16b}$$

with a negative sign for an outgoing ray and a positive sign for an incoming ray.

Equation (16) may be integrated analytically in two special cases. The first case is the class of isotropic cosmological models where $k^4 = 1/S(t)$ with $R = rS(t)$. The second case is where $M' = 0$ (empty space) and here we have

$$k^4 = \frac{\Gamma - \dot{R}}{1-2M/R} = \frac{1}{(1-2M/R)^{1/2}}\left(\frac{\Gamma - \dot{R}}{\Gamma + \dot{R}}\right)^{1/2}. \tag{17}$$

The first term on the right-hand side of (17) is the usual Schwarzschild red-shift factor, whereas the second is due to the motion. Equation (17) is most easily found by transforming from the usual Schwarzschild coordinates to co-moving coordinates by equation (8).

In general, however, it is difficult to integrate (16) and one must resort to numerical integration. Equation (12) was integrated numerically and the red shift estimated using

$$1+z = \tau_{\text{observed}}/\tau_{\text{emitted}} \tag{18}$$

where $\tau_{(r)}$ is the time interval between the arrival of two rays at coordinate radius r. The time interval between emission of the rays is assumed small.

The results for various forms of f and M are given in figures 1–3.

6. Conclusions

A general pressure-free system containing both an expanding region and a region that ultimately collapses has been considered. It has been found that light rays emitted from matter in the collapsing region which meet a closed trapped surface in the collapsing region eventually fall back into a singularity and so an event-horizon develops and at later times no light rays can pass from the collapsing to the expanding region. If, however, a ray does not meet a closed trapped surface then, except for the limiting case considered in § 4, it must escape and furthermore it eventually overtakes all expanding matter and so no event-horizons develop in the expanding region. These results correct those of Raychaudhuri (1966) and Som (1968).

Acknowledgments

I would like to thank Dr G. J. Whitrow for many useful discussions during the preparation of this paper, and the Science Research Council who supported this work with a Research Studentship.

References

BANERJEE, A., 1966 a, *Phys. Rev.*, **150**, 1086–8.
—— 1966 b, *Proc. Phys. Soc.*, **91**, 794–7.
BONDI, H., 1947, *Mon. Not. R. Astr. Soc.*, **107**, 410–25.
FAULKNER, J., HOYLE, F., and NARLIKAR, J. V., 1964, *Astrophys. J.*, **140**, 1100–5.
JAFFE, J., 1969, *Ann. Phys.*, **55**, 374–90.
LANDAU, L. D., and LIFSHITZ, E. M., 1962, *Classical Theory of Fields* (Oxford: Pergamon).
MISNER, C. W., and SHARP, D. H., 1964, *Phys. Rev.*, **136**, B571–6.
PENROSE, R., 1965, *Phys. Rev. Lett.*, **14**, 57–9.
RAYCHAUDHURI, A. K., 1966, *Proc. Phys. Soc.*, **88**, 545–7.
ROBSON, E. T., 1968, *Ph.D. Thesis*, University of Newcastle.
SCHWARZSCHILD, K., 1916, *Sber. Preuss. Akad. Wiss.*, **7**, 189.
SOM, M. M., 1968, *J. Phys. A: Gen. Phys.*, **1**, 393–8.
SYNGE, J. L., 1966, *Mon. Not. R. Astr. Soc.*, **131**, 463–6.